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PLANAR STANDING AND MARKING-TIME REGIMES OF A BIPEDAL WALKING DEVICE*

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A walking device standing on one leg, not fastened at its points of support, is considered. A study is made of how the device maintains equilibrium of its supporting leg by compensating oscillations of its body. Phase trajectories are analysed. Conditions are investigated under which one-way communication is maintained and discontinued while the device is moving. Marking-time regimes are constructed.

The problem of a standing walking device is interesting, first, as a problem in the dynamics of servosystems, and, second, as a limiting case of the problem of locomotion. Marking-time regimes may be used in constructing a model of space locomotion**. (**Beletskii V.V. and Golubitskaya M.D., Model problem of the dynamics of bipedal space locomotion. Preprint 194, Moscow, Inst. Prikl. Mat. Akad. Nauk SSSR, 1982).

*1. **Description of the model. Equations of the standing problem.*** We consider a bipedal walking device consisting of a heavy rigid body and a pair of identical weightless legs (Fig.1); each leg may consist of one or several segments. The legs are attached to the body of the device by double hinges at a point O. The device is assumed to be supported on one leg only. The leg is in contact with the supporting surface at a single point S, at which there acts a reaction force $\mathbf{R}_S$; communication with the surface is one-way (non-restoring). At the hinge O a controlling torque $\mathbf{Q}$ acts on the body and a torque $-\mathbf{Q}$ on the leg.

We assume that the supporting leg is maintained in equilibrium - the suspension point O and support point S remain fixed. The system is subject to feedback: the motion of the body is designed to maintain equilibrium of the supporting leg.

We shall consider planar regimes of motion. Fix a coordinate frame $NXYZ$ (Fig.1), where N is the origin and the NZ axis is directed vertically upward. The support and suspension points are assumed to lie in the NYZ plane: $S=(0, d, 0)$, where $d=\text{const}$, $d>0$, is the horizontal displacement of the support, and $O=(0,0,H)$, where $H=\text{const}$, $H>0$, is the height of the suspension point of the legs. It is assumed that the body does not spin; the centre of mass C moves in the NYZ plane.

We adopt the following notation: θ is the angle between the NZ axis and the vector $\mathbf{OC}$ in the positively oriented system $NXYZ$ (Fig.1), t is the time, g is the acceleration of free

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fall, M is the mass of the body, J, J_C are the moments of inertia of the body about the points O and C, and $\rho = OC$ is the distance from O to the centre of mass.

We introduce the following non-dimensional variables

$$\tau = \sqrt{\frac{M\rho g}{J}}\,t,\quad j = \frac{J}{M\rho^2} = 1 + \frac{J_C}{M\rho^2}$$
$$\delta = \frac{d}{\rho},\quad h = \frac{H}{\rho},\quad \mathrm{q} = \frac{\mathrm{Q}}{Mg\rho},\quad \mathrm{R} = \frac{\mathrm{R}_S}{Mg} \tag{1.1}$$

The derivative of θ with respect to non-dimensional time τ is denoted by θ'.

Under our assumptions, the equation of motion of the body in the variables (1.1) /1/ may be reduced to the form

$$\Phi(\theta)\,\theta'' + \frac{d\Phi(\theta)}{d\theta}\,\theta'^2 = j(\delta + \sin\theta) \tag{1.2}$$

$$\Phi(\theta) = j + h\cos\theta + \delta\sin\theta \tag{1.3}$$

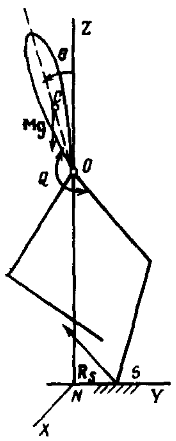

Fig.1

Equation (1.2) has a first integral /1/ (c is a constant of integration)

$$\theta' = \pm\frac{\sqrt{F(\theta) + c}}{\Phi(\theta)},\quad F(\theta) = \frac{2}{j}\int_0^\theta \Phi(\theta)(\delta + \sin\theta)\,d\theta \tag{1.4}$$

The expression for the controlling torque in the variables (1.1) /2/ is

$$q = -\delta + \frac{1}{j}\left(\theta'^2\frac{d\Phi}{d\theta} - \theta''\frac{d^2\Phi}{d\theta^2}\right) \tag{1.5}$$

The formula for the vertical reaction /2/ may be transformed as follows:

$$R_Z = 1 - \frac{1}{j}(\theta'^2\cos\theta + \theta''\sin\theta) \tag{1.6}$$

Using (1.2), we eliminate θ'' from (1.5), (1.6):

$$q = \sin\theta - [j(\delta + \sin\theta) - \theta'^2 d\Phi/d\theta]/\Phi \tag{1.7}$$

$$R_Z = [j(j-1) + j\cos\theta\,(h + \cos\theta) + \theta'^2(h + j\cos\theta)]/(j\Phi) \tag{1.8}$$

Motion of the system with non-restoring communication is realizable if the vertical reaction of the support is positive: $R_Z > 0$. In that case the control (1.7) leaves the supporting leg fixed and induces a compensating motion of the body, as described by integral (1.4).

2. *Analysis of phase trajectories for a device standing on a fastened leg.*

Let us study the motion of the system on the assumption that the leg is fastened to the surface at S by a hinge. A partial investigation may be found in /1/; here we present a more rigorous analysis.

Eq.(1.2) involves the following parameters: j is the moment of inertia of the body, δ is the support displacement, and h is the height of the suspension point of the legs. The values of the parameters j, δ, h determine the form of the phase portrait of Eq.(1.2). Define an additional parameter $r = \sqrt{\delta^2 + h^2}$ - the distance from the support point to the suspension point (in the case of a one-segmented leg, r is simply the length of the leg). To fix our ideas, we assume that $\theta \in [-2\pi, 0]$.

The main properties of the motion are as follows.

If $r > j$, there exist critical values $\theta = \theta^*$ at which the denominator of (1.7) vanishes: $\Phi(\theta^*) = j + h\cos\theta^* + \delta\sin\theta^* = 0$. This equation has two solutions:

$$\sin\theta_{1,2}^* = (-j\delta \pm h\sqrt{r^2 - j^2})/r^2,\quad \cos\theta_{1,2}^* = (-jh \mp \delta\sqrt{r^2 - j^2})/r^2$$
$$\theta_1^* \in [-3\pi/2, -\pi/2],\quad \theta_2^* \in [-\pi, 0] \tag{2.1}$$

If the numerator in (1.7) does not vanish, then at $\theta = \theta_{1,2}^*$ the controlling torque q is infinitely large, and it is almost always impossible to keep the point O fixed; in other words, at positions θ_1^*, θ_2^* the feedback loop is broken. This phenomenon was termed in /1, 2/ "parametric shock", since in the critical positions the absolute value of the supporting reaction increases without limit in absolute value. It is obvious from (1.4) that the angular velocity at $\theta = \theta_{1,2}^*$ is also infinitely large, and the phase curves experience discontinuities.

When $\delta < 1$ Eq.(1.2) has two steady-state solutions:

$$\sin\theta_{1,2}^{\circ} = -\delta, \quad \cos\theta_{1,2}^{\circ} = \mp\sqrt{1-\delta^2}$$
$$\theta_1^{\circ} \in [-\pi, -\pi/2], \quad \theta_2^{\circ} \in [-\pi/2, 0] \tag{2.2}$$

Linearizing (1.2) in the neighbourhood of the point θ_i° $(i = 1, 2)$, we obtain

$$\varphi'' = \frac{j\cos\theta_i^{\circ}}{\Phi(\theta_i^{\circ})}\varphi, \quad \varphi = \theta - \theta_i^{\circ}, \quad i = 1, 2 \tag{2.3}$$

In the upper equilibrium position $\Phi(\theta_2^{\circ}) = j - \delta^2 + h\sqrt{1-\delta^2} > 0$, since $j \geqslant 1, \delta < 1$. It follows from (2.2), (2.3) that the upper equilibrium is always unstable.

The lower equilibrium position $(\cos\theta_1^{\circ} < 0)$ is stable if

$$\Phi(\theta_1^{\circ}) > 0 \tag{2.4}$$

and unstable if $\Phi(\theta_1^{\circ}) \leqslant 0$. The function $\Phi(\theta)$ takes negative values only in the case $r > j$, if $\theta \in (\theta_1^*, \theta_2^*)$; consequently, the stability of the stationary point θ_2° depends on its position relative to the critical points. Comparing $\sin\theta_1^{\circ} = -\delta$ and $\sin\theta_{1,2}^*$, we obtain

$$\sin\theta_{1,2}^* + \delta = [\pm h\sqrt{r^2 - j} + \delta(r^2 - j^2)]/r^2 \tag{2.5}$$

By (2.5), it is always true that $\theta_1^* < \theta_1^{\circ}$ (the discussion /1/ erroneously referred to the case $\theta_1^* > \theta_1^{\circ}$), and both cases $\theta_1^{\circ} < \theta_2^*$ and $\theta_1^{\circ} \geqslant \theta_2^*$ are possible. The equilibrium position θ_1° is unstable if and only if it lies between the critical positions; more precisely, if and only if $\theta_1^{\circ} \in (\theta_1^*, \theta_2^*]$. If $r < j$ (there are no critical positions) or $\theta_1^{\circ} > \theta_2^*$, the lower equilibrium position is stable. Written explicitly, the stability condition (2.4) is

$$h < (j - \delta^2)/\sqrt{1-\delta^2} \tag{2.6}$$

Let c^* denote the values of the integration constant for which the numerator in (1.7) vanishes at the same points θ^* as the denominator. The corresponding phase curves (1.4) are denoted by $\theta'(c^*, \theta)$. By (1.7),

$$\theta'^2(c_i^*, \theta_i^*) = j(\delta + \sin\theta_i^*)\left(\frac{d\Phi}{d\theta}(\theta_i^*)\right)^{-1}, \quad i = 1, 2 \tag{2.7}$$

Recall that $\theta_{1,2}^*$ are the roots of the function $\Phi(\theta)$, at which it changes sign: $d\Phi(\theta_1^*)/d\theta < 0$, $d\Phi(\theta_2^*)/d\theta > 0$. It is obvious from (2.5) that $\delta + \sin\theta_1^* > 0$. Consequently, at $\theta = \theta_1^*$ the right-hand side of (2.7) is negative and the real value of the angular velocity on the phase curve $\theta'(c_1^*, \theta)$ is undefined, so the feedback loop is broken.

It follows from (2.5) that $\delta + \sin\theta_2^* > 0$ if

$$h^2(1 - \delta^2) > (j - \delta^2)^2 \tag{2.8}$$

Under these conditions the right-hand side of (2.7) is positive, the value of the angular velocity at $\theta = \theta_2^*$ on the phase curve $\theta'(c_2^*, \theta)$ is finite:

$$\theta'(c_2^*, \theta_2^*) = \pm\left\{\frac{j[\delta(r^2 - j) - h\sqrt{r^2 - j^2}]}{r^2\sqrt{r^2 - j^2}}\right\}^{1/2} \tag{2.9}$$

Under conditions (2.8), there exists a phase curve $\theta'(c_2^*, \theta)$ which is continuous at the critical point θ_2^*. It can be shown that at θ_2^* on the phase curve the acceleration $\theta''(c_2^*, \theta_2^*) = j\delta(1 - j)/(3\sqrt{r^2 - j^2})$ of the body is continuous and finite, as are the control and the reaction force; in other words, the feedback is maintained.

Comparing conditions (2.6), (2.8), we note that if $\delta < 1$, $r > j$, i.e., the critical points exist and are stationary, then passage through the critical point θ_2^* at a finite angular velocity is possible if and only if the lower equilibrium position θ_1° is stable. And conversely. The phase curves are all discontinuous at θ_2^* if and only if there exists an unstable stationary point θ_1° $(\cos\theta_1^{\circ} < 0)$.

This analysis shows that the surfaces $\delta = 1$, $h = (j - \delta^2)/\sqrt{1-\delta^2}$, $\sqrt{\delta^2 + h^2} = j$ $(r = j)$ divide the parameter space (j, δ, h) into subspaces, each corresponding to a qualitatively distinct phase portrait. The projection of this partition on a plane $j = \text{const}$ is shown in

Fig.2. Fig.3, a-e, illustrates the corresponding phase portraits.

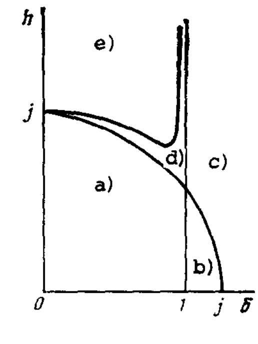

Fig.2

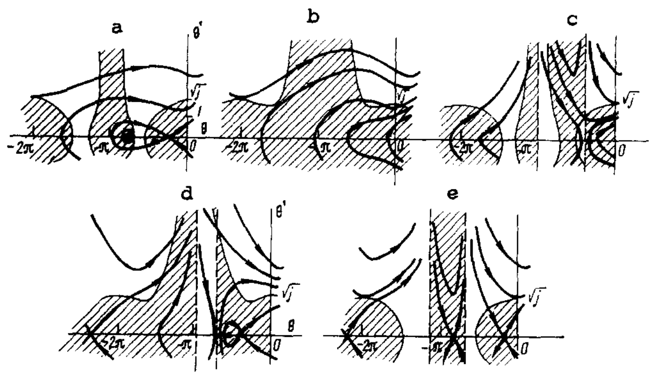

Fig.3

The phase portraits will now be described in detail.

a) $r < j$, $\delta < 1$ (Fig.2). There are two stationary points: a stable one θ_1° and an unstable one θ_2°. There are no critical points. The phase portrait for this case is shown in Fig.3a. In the neighbourhood of the stable equilibrium position the body has a region of oscillatory motions. All phase trajectories outside this region represent unlimited unwinding motion of the body.

b) $r < j$, $\delta > 1$ (Fig.2). There are no stationary points, the region of oscillatory motions disappears. No points of discontinuity. All motions of the device reduce to unlimited unwinding of the body (Fig.3b).

c) $r > j$, $\delta > 1$. No stationary points, two critical points θ_1^* and θ_2^*. There is a phase curve that is continuous at θ_2^* (Fig.3c). All phase trajectories are discontinuous: having starting to move from any position, the body will assume an infinite angular velocity within a finite time. The behaviour of the phase trajectories after "passing through" the critical points can be described in a purely formal manner as follows. The (θ, θ') plane is divided by the phase trajectory $\theta'(c_2^*, \theta)$ into an inner region (discontinuous at $\theta = \theta_1^*$) and an outer region. The trajectories of the inner region resemble the closed trajectories of Fig.3a - a kind of "oscillation" about the critical point θ_1^*. The other trajectories are analogous to the unlimited unwinding trajectories of Fig.3a (in each complete revolution the body must overcome two critical positions).

d) $r > j$, $\delta < 1$, $h < (j - \delta^2)/\sqrt{1 - \delta^2}$ (Fig.2). Two stationary points θ_1°, θ_2°, one of which is stable, and two critical points θ_1^*, θ_2^*, through one of which there is a continuous phase curve. The phase portrait is shown in Fig.3d. The description in /1/ of this phase portrait and that corresponding to the case $r = j$, $\delta = 1$ is erroneous.

e) $h > (j - \delta^2)/\sqrt{1 - \delta^2}$. Two unstable stationary points, two critical points, forming the following sequence in the interval $[-2\pi, 0]$: $\theta_1^* < \theta_1^\circ < \theta_2^* < \theta_2^\circ$ (Fig.3e). All trajectories are discontinuous at θ_2^*. All motions break the feedback loop.

3. Regions of coupled motion. The phase portraits of Fig.3 describe the motion of the system with two-way communication between the support point and the surface. Throughout the sequel we shall assume that the leg is not attached at the point S. Motion with non-restoring communication will take place along the phase trajectories of Fig.3 if $R_Z > 0$ throughout.

Putting $\theta = \theta_{1,2}^{\circ}$ in (1.6), we see that at the stationary points $R_Z = 1$. Since R_Z is continuous, it follow that in sufficiently small neighbourhoods of these points the vertical reaction is positive. In particular, if the system is in the stable equilibrium position $\theta = \theta_1^{\circ}$, or performing sufficiently small oscillations in the vicinity of the latter, the non-restoring communication of the leg with the surface will be maintained: the device will remain standing for as long as desired on one leg, balanced by the body about the lower (non-vertical) position θ_1°.

Determination of the sign of R_Z in all other (non-oscillatory) motions of the system requires further investigation.

We will write the value (1.8) of R_Z along the phase trajectories as follows:

$$R_Z = \left[(f(\theta) - \theta'^2)\left(\Phi\cos\theta - \frac{d\Phi}{d\theta}\sin\theta\right)\right] \Big/ (j\Phi) \tag{3.1}$$

$$f(\theta) = j\left[\Phi - (\delta + \sin\theta)\sin\theta\right] \Big/ \left(\Phi\cos\theta - \frac{d\Phi}{d\theta}\sin\theta\right) \tag{3.2}$$

It is clear from (3.1) that R_Z changes sign when $f(\theta) = \theta'^2$ or $\Phi(\theta) = 0$. In other words, in the (θ, θ') plane the curve

$$\theta' = \varkappa(\theta) = \pm\sqrt{f(\theta)} \tag{3.3}$$

and the critical lines $\theta = \theta_{1,2}^*$, wherever defined, will be the boundaries of regions of coupled motion (regions in which the relation $R_Z > 0$ is maintained).

By (3.2) and (3.7),

$$f(\theta_i^*) = j(\delta + \sin\theta_i^*) \Big/ \frac{d\Phi}{d\theta}(\theta_i^*) = \theta_i'^2(c_i^*, \theta_i^*), \quad i = 1, 2$$

Consequently, $f(\theta_1^*) < 0$ for any j, δ, h, i.e., the curve $\varkappa(\theta) = \pm\sqrt{f(\theta)}$ has no points in common with the stright line $\theta = \theta_1^*$. If condition (2.8) holds, then $f(\theta_2^*) > 0$, otherwise $f(\theta_2^*) < 0$. This means that the curve $\varkappa(\theta)$ cuts the critical line $\theta = \theta_2^*$ if and only if there is a continuous phase trajectory $\theta'(c_2^*, \theta)$ passing through the line $\theta = \theta_2^*$. Both curves - $\theta'(c_2^*, \theta)$ and $\varkappa(\theta)$ - intersect the straight line $\theta = \theta_2^*$ at one point.

Substitute expression (1.3) for $\Phi(\theta)$ into (3.2):

$$f(\theta) = j(\cos^2\theta + h\cos\theta + j - 1)/(h + j\cos\theta) \tag{3.4}$$

Fig.4 shows the types of behaviour of the curve $\varkappa(\theta)$ for different values of j, h.

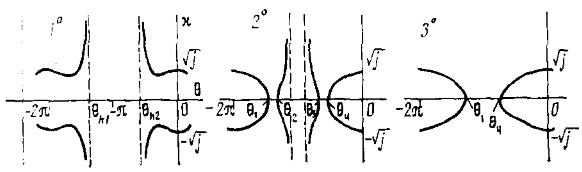

Fig.4

These different types will be described for $j \leqslant 2$ assuming, to fix our ideas, that $\theta \in [-2\pi, 0]$.

1°. $0 < h < 2\sqrt{j-1}$. The function $f(\theta)$ has no roots. At $\theta = \theta_{h1} = -2\pi - \theta_{h2}$ or $\theta = \theta_{h2} = -\arccos(-h/j)$ the denominator of (3.4) vanishes, and the function $f(\theta)$ changes sign. The curve $\varkappa(\theta)$ is defined in the intervals $[-2\pi, \theta_{h1}]$ and $[\theta_{h2}, 0]$ (Fig.4,1°).

2°. $2\sqrt{j-1} < h < j$. The function $f(\theta)$ is discontinuous at points θ_{h1}, θ_{h2} and has four roots in the interval $[-2\pi, 0]$: $\theta_1 = -2\pi - \theta_4, \theta_2 = -2\pi - \theta_3, \theta_{3,4} = -\arccos[{}^1/_2(\mp h - \sqrt{h^2 - 4j + 4})]$. The curve $\varkappa(\theta)$ is shown in Fig.4,2°.

3°. $h \geqslant j$. The function $f(\theta)$ is continuous and has roots θ_1, θ_4. The curve $\varkappa(\theta)$ is defined in the intervals $[-2\pi, \theta_1]$ and $[\theta_4, 0]$ (Fig.4,3°).

If $j \geqslant 2$ the curve $\varkappa(\theta)$ may have one of two distinct forms: 1°. $h < j$ (Fig.4,1°) and 3°. $h \geqslant j$ (Fig.4,3°).

We remark moreover that

$$\varkappa(0) = \pm\sqrt{j}, \quad \frac{d\varkappa}{d\theta}(0) = 0 \tag{3.5}$$

Appealing to the above analysis, we can determine the regions in the phase plane where $R_Z > 0$ - the hatched regions in the phase portraits of Fig.3. Their structure is determined by the values of the parameters j, δ, h and depends, first, on the relative positions of the curves $\varkappa(\theta)$, $\theta = \theta_{1,2}^*$ and, second, on the form of the curve $\varkappa(\theta)$.

The structure of the regions of coupled motion is simplest when $r < j$ (no critical positions; Fig.3a, b). R_Z has the same sign as the numerator in (3.1): $R_Z > 0$ if

$$(f(\theta) - \theta'^2)(h + j\cos\theta) > 0 \tag{3.6}$$

This inequality will hold if $\theta \in [\theta_{h1}, \theta_{h2}]$ or $\theta'^2 < f(\theta)$. The structure of the regions of coupled motion is readily understood by comparing Fig.3b with Fig.4, 1° and Fig.3a with Fig.4, 2°.

If $r \geqslant j$, the regions of $R_Z > 0$ for $\theta \in [\theta_1^*, \theta_2^*]$ are defined by the inequality inverse to (3.6), and for $\theta \notin [\theta_1^*, \theta_2^*]$ by condition (3.6). Examples are shown in Fig.3c-e.

We will now show how the region of coupled motion changes as the parameters δ, h vary in the plane $j = \text{const}$. Fig.5 is a section of (j, δ, h) space by such a plane, in Fig.5,1 for $j < 2$ and in Fig.5,2 for $j \geqslant 2$. The solid lines bound subspaces (strips 1-3° corresponding to the different forms of the curve $\varkappa(\theta)$ (Fig.4). The dashed curves bound sectors a-c corresponding to the different types of phase portrait (Fig.3). The intersection of one of the strips 1-3° with a sector a-c will be denoted by the appropriate digit and letter.

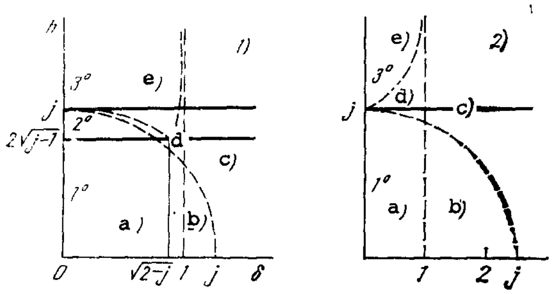

Fig.5

It is obvious from (3.4) that if $j = \text{const}, h = \text{const}$ the form of the curve $\varkappa(\theta)$ remains unchanged. We describe the evolution of the regions of coupled motion along the straight lines $h = \text{const}$.

To fix our ideas, we assume that $\theta \in [-2\pi, 0]$.

1°. $h = h_1 = \text{const}$; $0 < h_1 < 2\sqrt{j-1}$, $j < 2$ or $0 < h_1 < j$, $j \geqslant 2$ (strip 1° in Fig.5, Fig.4, 1° for the curve $\varkappa(\theta)$). The straight line $h = h_1$ intersects either sectors 1°a, 1°b, 1°c or 1°a, 1°d, 1°c (Fig.5).

1°a, 1°b $(\delta < \sqrt{j^2 - h^2})$. The regions $R_Z > 0$ corresponding to these sectors are exactly the same. An example for 1°b is shown in Fig.3b.

1°c, 1°d $(\delta \geqslant \sqrt{j^2 - h^2})$. At $\delta = \sqrt{j^2 - h^2}$ (on the boundary of sectors 1°b-1°c or 1°a-1°d) a critical line $\theta = \theta_1^* = \theta_2^*$ appears in the phase plane: it coincides with the right vertical asymptote of the function $f(\theta)$: $\theta_1^* = \theta_2^* = \theta_{h2}$. As δ increases from $\sqrt{j^2 - h^2}$ to $+\infty$, the line $\theta = \theta_1^*$ moves left from $\theta = \theta_{h2}$ to $\theta = -\pi$, and the line $\theta = \theta_2^*$ right from $\theta = \theta_{h2}$ to $\theta = 0$ (Fig.3d). The line $\theta = \theta_1^*$ does not intersect $\varkappa(\theta)$ and lies between the asmptotes: $\theta_{h1} < \theta_1^* < \theta_{h2}$, but the line $\theta = \theta_2^*$ cuts the curve and lies to the right of the asymptotes: $\theta_2^* > \theta_{h2}$. As the parameter δ goes through $\delta = 1$ the type of phase portrait changes, but no change occurs in the general shape of the region $R_Z > 0$. The region $R_Z > 0$ corresponding to sector 1°d is shown in Fig.3d. In order to construct the region $R_Z > 0$ for sector 1°c one must plot a similar hatched region in the phase portrait of Fig.3c.

2°. $h = h_2 = \text{const}$; $2\sqrt{j-1} < h_2 < j$, $j < 2$ (strip 2° in Fig.5,1). Fig.4,2° for the curve $\varkappa(\theta)$.

The basic difference between the cases $j < 2$ and $j \geqslant 2$ (Fig.5,1, Fig.5,2) is that for $j = \text{const}$, $j < 2$ the function

$$h(\delta) = (j - \delta^2)/\sqrt{1 - \delta^2} \tag{3.7}$$

has a minimum $\delta = \sqrt{2 - j}$, $h_{\min} = 2\sqrt{1 - j}$. The existence of this minimum or, more precisely, the fact that the curve (3.7) may lie below the straight line $h = j$, implies the existence of a curve $\varkappa(\theta)$ (Fig.4,2°) and regions of coupled motion characteristic for strip 2° but not existing in the case $j \geqslant 2$ (Fig.3a,c, Fig.6 1°, 2°).

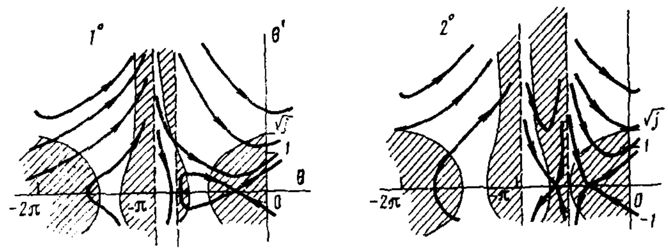

Fig.6

All the regions $R_Z > 0$ corresponding to the line $h = h_2$ are multiply connected. If $\delta \leqslant 1$ (sectors 2°a, 2°d, 2°e) the motion has stationary points. It can be shown that the unstable stationary point $\theta_2{}^{\circ}$ lies to the right of all roots of the function $f(\theta)$ (Fig. 4, 2°): $\theta_2{}^{\circ} > \theta_4$, while the following inequalities hold for the point $\theta_1{}^{\circ}$: if the point is stable $(h < (j - \delta^2)/\sqrt{1-\delta^2})$ and $\delta < \sqrt{2-j}$, then $\theta_1{}^{\circ} < \theta_3$; if the point is unstable $(h \geqslant (j-\delta^2)/\sqrt{1-\delta^2})$, then $\theta_3 \leqslant \theta_1{}^{\circ} \leqslant \theta_4$; if the point is stable and $\delta > \sqrt{2-j}$, then $\theta_1{}^{\circ} > \theta_4$.

Let δ_1, δ_2 denote the points at which the line $h = \text{const}$ cuts the curve (3.7) : $\delta_1 < \sqrt{2-j}$, $\delta_2 > \sqrt{2-j}$.

We will list the sectors cut out by the line $h = h_2$ and describe the corresponding regions $R_Z > 0$.

2°a $(0 < \delta < \sqrt{j^2 - h^2})$. There are not critical lines. The stable stationary point $\theta_1{}^{\circ}$ lies to the left of the root θ_3 (Fig.3a).

2°d $(\sqrt{j^2 - h^2} < \delta \leqslant \delta_1)$. Two critical lines, one of which $(\theta = \theta_2{}^*)$ cuts the curve $\varkappa(\theta)$: $\theta_{h2} < \theta_2{}^* < \theta_3$ (Fig.6,1°). The stable stationary point is such that $\theta_2{}^* < \theta_1{}^{\circ} < \theta_3$. As $\delta \to \delta_1$ we have $\theta_2{}^* \to \theta_3$, $\theta_1{}^{\circ} \to \theta_3$; on the boundary of sectors 2°d, 2°e $(\delta = \delta_1)$ all three points coincide: $\theta_2{}^* = \theta_1{}^{\circ} = \theta_3$.

2°e $(\delta_1 < \delta < \delta_2)$. The region $R_Z > 0$ is shown in Fig. 6, 2°. The critical line $\theta = \theta_2{}^*$ has no points in common either with the phase trajectories or with the curve $\varkappa(\theta)$: $\theta_3 < \theta_2{}^* < \theta_4$. As δ increases from $\delta = \delta_1$ to $\delta = \delta_2$ the line $\theta = \theta_2{}^*$ moves monotonically from position $\theta = \theta_3$ to position $\theta = \theta_4$. The position of the unstable stationary point $\theta_1{}^{\circ}$ is $\theta_3 < \theta_1{}^{\circ} < \theta_4$. As $\delta \to \delta_2$ we have $\theta_1{}^{\circ} \to \theta_4$, $\theta_2{}^* \to \theta_4$; on the boundary of sectors 2 e, 2°d $(\delta = \delta_2)$ all three points coincide: $\theta_1{}^{\circ} = \theta_2{}^* = \theta_4$.

2°d $(\delta_2 \leqslant \delta \leqslant 1)$; 2°c $(\delta > 1)$. the line $\theta = \theta_2{}^*$ intersects the curve $\varkappa(\theta)$ to the right of the root θ_4 (Fig.3c). If $\delta_2 < \delta \leqslant 1$, the stable stationary point lies to the right of the critical point $\theta_2{}^*$. When $\delta > 1$ the stationary points disappear, but the structure of the regions $R_Z > 0$ remains essentially the same. An example of the region $R_Z > 0$ for sector 2°c is shown in Fig.3c.

3°. $h = h_3 = \text{const}$; $h_3 \geqslant j$ (strip 3° in Fig.5, Fig.4, 3° for the curve $\varkappa(\theta)$). The curve $\varkappa(\theta)$ is defined in the intervals $[-2\pi, \theta_1]$ and $[\theta_4, 0]$. Critical lines exist for any values of δ.

3°e $(0 < \delta \leqslant \delta_2)$. The region $R_Z > 0$ is shown in Fig.3e. At $\delta = 0$ the lines $\theta = \theta_{1,2}{}^*$ are symmetric about $-\pi$: $\theta_1{}^* = -2\pi - \theta_2{}^*$, $\theta_2{}^* = -\arccos(-j/h)$, $\theta_2 < \theta_1{}^* < \theta_2{}^* < \theta_4$. As δ increases monotonically both lines move to the right; in particular, at $\delta = \delta_2$ we have $\theta_2{}^* = \theta_4$.

3°d $(\delta_2 < \delta \leqslant 1)$, 3°c $(\delta > 1)$. The region $R_Z > 0$ for sector 3°c is shown in Fig.7. As $\delta \to +\infty$ we have $\theta_1{}^* \to -\pi$, $\theta_2{}^* \to 0$.

4. Protracted unwinding. It is obvious from the examples of Figs.3, 6, and 7 that all non-oscillatory motions - and some oscillatory ones - of the device break the feedback loop.

If the body lies at time $\tau = 0$ between the critical angles, $\theta(0) \in (\theta_1{}^*, \theta_2{}^*)$, it must assume a critical position within a finite interval of time - the point O cannot be held fixed at all subsequent times. But in addition R_Z changes sign at the critical positions, so that the support point cannot be held fixed either (Figs.3c-e, Figs. 6, 7).

In other cases of non-oscillatory motion $(\theta(0) \notin [\theta_1{}^*, \theta_2{}^*])$ the vertical reaction will decrease to zero within a finite time and then become negative (Figs.3, 6, 7).

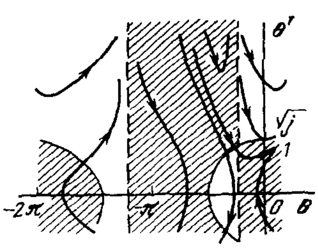

Fig.7

At the instant R_Z vanishes, the leg is detached from the surface and the device begins to fall. The angle through which the body turns while falling depends on the extent and structure of the region of coupled motion. If the region $R_Z > 0$ is simply connected Fig.3b), the angle may be quite large. The conditions for the region to be simply connected are: $r < j$; for $j < 2, h < 2\sqrt{j-1}$. Recalling that $j = j_C + 1$ and $j_C = J_C/(M\rho^2) = K^2/\rho^2$, where j_C is the central moment of inertia of the body and K is the central radius of inertia, we can write these conditions in the dimensional variables:

$$d^2 + H^2 < K^2 \tag{4.1}$$

If $H < 2K$, then $K < \rho$.

The phase trajectories within a simply connected region $R_Z > 0$ are trajectories either of bounded oscillations or of unlimited unwinding. It is interesting to consider a fairly long unwinding motion of the body, say lasting for several complete revolutions. It follows from (4.1) that this is possible if the support displacement d and height H are sufficiently small or if the device is designed with a sufficiently large central radius of inertia K. As $j \to \infty$ the region of coupled motion covers the entire phase plane (Fig.3b). This is possible only if $K \to \infty$ (ρ fixed) or, more realistically, if $\rho \to 0$ (K fixed). Only in this last case, which has been discussed previously /3/, can the body unwind for an infinitely long time.

All the standing regimes described above may be regarded as limiting cases of locomotive regimes when the length of the step and the velocity tend to zero. They may be used as "generating" motions in the solution of various locomotion problems.

5. *Marking-time regime.* If j is finite (K finite, $\rho \neq 0$), any motion of the body through the upper vertical position makes the device fall (Figs.3, 6, 7). In some cases falling can be prevented by allowing the point of support to change and switching the support leg in time, say when $\theta = 0$. Repetition of this process results in the device "marking time", so to speak. In such regimes the body of the device will oscillate about the position $\theta = 0$.

Marking time on a single support leg may be accomplished as follows. Suppose that the support point is situated on the NY axis: $S(0, y_S, 0)$. Fig.3a-d shows the phase portraits of the body's motion when $y_S = \delta$, $\delta > 0$ (the device is standing on its left leg). When $y_S = -\delta$ (standing on the right leg) the phase portraits are symmetric about the axis $\theta = 0$ to those shown. Consider the phase portraits of the composite motion defined by $\theta < 0$ for $y_S = \delta$, $\theta > 0$ for $y_S = -\delta$. In other words, the coordinate of the support point obeys the law: $y_S = \delta \operatorname{sign} \theta$ (for the support leg to be switched at $\theta = 0$, the control must be turned off at the hinge O of the support leg when $\theta = 0$, and that of the other leg turned on at its hinge O). Throughout the sequel we shall consider only such composite phase portraits.

It is obvious from Fig.3 that in all five types of composite phase portrait there must exist closed phase trajectories - the trajectories of the oscillations performed by the body about its upper vertical position. We shall refer to the part of the composite phase plane in which all phase trajectories are closed trajectories of periodic motions about $\theta = 0$ as the oscillatory region. By definition, the oscillatory region does not contain critical or stationary points. Let $\theta'(\pm c_\gamma, 0)$ denote the phase trajectories passing through the points $(\pm\theta_\gamma, 0)$ of the phase plane, where

$$\theta_\gamma = \begin{cases} \theta_2^\circ & \text{for } \delta \leqslant 1 \\ -\infty & \text{for } \delta > 1,\ r < j \\ \theta_2^* & \text{for } \delta > 1,\ r \geqslant j \end{cases}$$

The trajectories $\theta'(\pm c_\gamma, \theta)$ are the boundaries of the oscillatory region for $\theta \in (\theta_\gamma, -\theta_\gamma)$; all the types of oscillatory region are shown in Fig.8. If $\delta > 1$, $r < j$, the oscillatory region is unbounded (Fig.8, 2°); if $\delta > 1$, $r \geqslant j$, the boundaries are the trajectories $\theta'(\pm c_2^*, \theta)$ and the straight lines $\theta = \pm\theta_2^*$ (Fig.8, 3°). The oscillatory region is generally more extensive than the region $R_Z > 0$ (Figs.8, 2°, 3°).

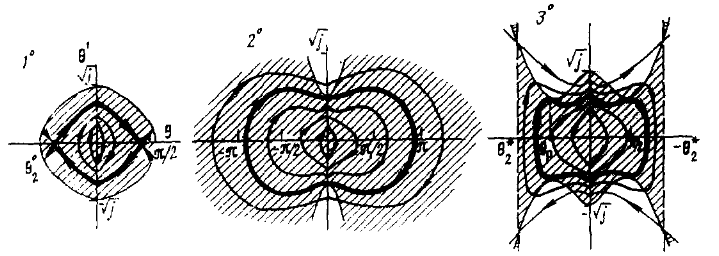

Fig.8

Within the intersection of the oscillatory region and the region of coupled motion there is a region of coupled oscillations, which we shall call the controllability region. The boundaries of the controllability regions are shown in Fig.8 by thick curves. All trajectories in these regions correspond to periodic motion of the body while marking time. It is assumed that during the first step the device is supported on its left leg: $y_S = \delta$, $\theta < 0$, during its second step it stands on its right leg: $y_S = -\delta, \theta > 0$; and so on. If each step takes time τ_k, the body will oscillate with period $2\tau_k$, inclining during each step in the direction of the supporting leg.

The basic characteristics of these oscillations depend on the shape of the controllability region and are determined by the values of the parameters j, δ, h. The shapes of the controllability regions for different parameter values may be described as follows.

The phase trajectories of the oscillations can leave the region of coupled motion only through the boundary $\varkappa(\theta)$ (Fig.8). The direction of the trajectories at different points of the θ, θ' plane is determined by the vector field of the control (1.2):

$$\mathbf{w} = \{w_\theta, w_{\theta'}\}$$
$$w_\theta = \theta', \ w_{\theta'} = [j(\delta + \sin\theta) - (d\Phi/d\theta)\theta'^2]/\Phi(\theta) \tag{5.1}$$

It follows from (5.1) that if $\theta'^2 = j(\delta + \sin\theta)(d\Phi/d\theta)^{-1}$ the phase trajectories have horizontal tangents. The condition

$$\theta' = \eta(\theta) = \pm [j(\delta + \sin\theta)(d\Phi/d\theta)^{-1}]^{1/2} \tag{5.2}$$

defines a continuous curve $\eta(\theta)$ in the phase plane. It can be proved that if $\theta \in (\theta_\gamma, -\theta_\gamma)$ then $|\eta(\theta)| < |\varkappa(\theta)|$, and if $\delta \leqslant 1$ then $|\theta'(c_\gamma, \theta)| < |\eta(\theta)|$. Consequently, $|\theta'(c_\gamma, \theta)| < |\varkappa(\theta)|$, i.e., the oscillatory region lies within the region of coupled motion, the controllability region coincides with the oscillatory region.

When $\delta > 1$ the boundary of the region of coupled motion may lie within the oscillatory region.

In a neighbourhood of the curve $\varkappa(\theta)$, consider the function $G(\theta, \theta') = f(\theta) - \theta'^2$. Within the region of coupled motion $G(\theta, \theta') > 0$, outside it $G(\theta, \theta') \leqslant 0$. Let $\Delta(\theta, \theta')$ denote the derivative of $G(\theta, \theta')$ in the direction of the vector field (5.1): $\Delta(\theta, \theta') = (\mathbf{w} \cdot \operatorname{grad} G)$. Omitting the intermediate steps, we write the final expression for $\Delta(\theta, \theta')$ on the curve $\varkappa(\theta)$:

$$\Delta_\varkappa = \Delta(\theta, \varkappa(\theta)) = -2j\theta' p(\cos\theta)\sin\theta/(h + j\cos\theta)^2$$
$$p(\cos\theta) = 3j\cos^2\theta + 2h(2 + j)\cos\theta + 3h^2 - j(j-1) \tag{5.3}$$

The value of $\operatorname{sign}\Delta_\varkappa$ characterizes the behaviour of the phase trajectories on the boundary of the region $R_Z > 0$: if $\Delta_\varkappa > 0$, the function $G(\theta, \theta')$ increases along the phase trajectory, the trajectory enters the region $R_Z > 0$; but if $\Delta_\varkappa < 0$ then $G(\theta, \theta')$ decreases and the trajectory leaves the region. The direction of the phase trajectories on the boundary $\varkappa(\theta)$ determine the shape of the controllability region when $\delta > 1$.

Let $\theta \in [-\pi, 0], \theta' > 0$. In this part of the phase plane $\operatorname{sign}\Delta_\varkappa = \operatorname{sign}[p(\cos\theta)]$.

Analysis shows that if $h < (j-4)/3$, $j > 4$, then $p(\cos\theta) < 0$ for any θ; if $h > j$, then $p(\cos\theta) > 0$ for any θ. If $(j-4)/3 \leqslant h \leqslant j$ then $p(\cos\theta)$ has roots in the interval $[-\pi, 0]$, one of which - the largest - characterizes the shape of the controllability region for $\delta > 1$. Denote this root by θ_p. We have

$$\cos\theta_p = (-h(j+2) + \{(j-1)[h^2(j-4) + 3j^2]\}^{1/2})/(3j) \tag{5.4}$$

$dp(\cos\theta_p)/d\theta > 0$, and if $\theta \in (\theta_\gamma, 0]$ then $p(\cos\theta) > 0$. The function $p(\cos\theta)$ changes sign in the interval $(\theta_\gamma, 0)$ only if $\theta_p \in (\theta_\gamma, 0)$. This condition holds if $\delta > 1, r < j$. When $\delta > 1$, $r \geqslant j$ $(\theta_\gamma = \theta_2^*)$, further analysis is necessary.

Using the equation

$$\cos\theta_p = \cos\theta_2^* \tag{5.5}$$

one can determine for any $j, h, h < j$, a number $\delta_p(j, h)$ such that whenever $1 < \delta < \delta_p(j, h)$ we have $\theta_p \in (\theta_\gamma, 0)$, and whenever $\delta \geqslant \max\{1, \delta_q(j, h)\}$ we have $\theta_p \in (\theta_\gamma, 0)$. The right-hand side of Eq.(5.5) is defined by (5.4) and the left-hand side by (2.1).

The surface $\delta = 1, \delta = \delta_p(j, h), h = (j-4)/3$ (for $j > 4$) divides the parameter space (j, δ, h) into subspaces, each with its specific type of controllability region. Fig.9 is a projection of this partition on a plane $j = \text{const}$, $j > 4$. For examples of controllability regions, see Fig.8.

The different types of controllability region will now be described.

1°. $\delta \leqslant 1$. The controllability region coincides with the oscillatory region (Fig.8, 1°). The amplitudes of the oscillations are bounded by $|\theta_2^\circ| \leqslant \pi/2$, and the angular velocity by

$|\theta'(c_2^\circ, 0)| < \sqrt{j}$. The duration of the step is not bounded: the device may stand on one leg for a finite but for as long a time as desired if the amplitude of the oscillations is as close to $|\theta_2^\circ|$ as desired.

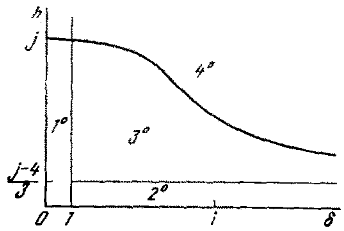

Fig.9

2°. $\delta > 1,\ h < (j-4)/3,\ j > 4$. For any θ we have $p(\cos\theta) < 0$ - if $\theta \in (-\pi, 0)$ the phase trajectories can only leave the region of coupled motion. The boundary of the controllability region is the trajectory $\theta'(\pm c_j, 0)$ passing through the points $(0, \pm\sqrt{j})$. An example for $r < j$ is illustrated in Fig.8,2.

On the boundary trajectory the maximum angular velocity for $\theta = 0$ is attained:

$$\max_{\delta, h, c} |\theta'(c, 0)| \sqrt{j} = |\theta'(c_j, 0)|$$

Note that when $\delta > 1$ the number $\sqrt{j}$ is a bound for $\theta'(c, \theta)$ only at the time the legs are switched $(\theta = 0)$; during the step $(\theta \neq 0)$ the angular velocity $\theta'(c, \theta)$ may exceed $\sqrt{j}$ (Fig.8,2°). Moreover, when $\delta > 1$ the step time and oscillation period are bounded.

It follows from (1.6) and (1.7) that in motion along a boundary trajectory $\theta'(\pm c_j, \theta)$ the support leg is switched under the action of a continuous control: when $\theta = 0$ $q = 0$; the support leg is detached and the other leg placed in position "softly": when $\theta = 0$ $R_Z = 0$.

3°. $h > (j-4)/3,\ 1 < \delta < \delta_p(j, h)$. The function $p(\cos\theta)$ has a root $\theta_p \in (\theta_\gamma, 0)$. If $\theta \in (-\pi, \theta_p)$ the oscillatory trajectories can only leave the region $R_Z > 0$, and if $\theta \in (\theta_p, 0)$ they can only enter it. The trajectory through the points $\pm\theta_p, \varkappa(\pm\theta_p)$ is tangent to $\varkappa(\theta)$. This trajectory is precisely the boundary of the controllability region. Fig.8, 3° gives an example for $r \geqslant j$.

4°. $h > (j-4)/3,\ \delta \geqslant \max\{1, \delta_p(j, h)\}$. When $\theta \in (\theta_\gamma, 0)$ we have $p(\cos\theta) > 0$. The oscillatory trajectories cannot leave the region $R_Z > 0$. The controllability region coincides with the oscillatory region. The amplitude of the oscillations may exceed $\pi/2$, e.g., when $\delta_p(j, h) < \delta < j$ we have $|\theta_\gamma| = |\theta_2^*| > \pi/2$.

The marking-time regimes just constructed may be regarded as a limiting case of space locomotion when the length of the step and velocity tend to zero. Imposing an additional condition on the system parameters $(j = 1,\ j_C = 0)$, one can construct a model of space locomotion in which these regimes serve as the basic components*.(*see the previous footnote). The motion of the device in this model takes place along the NX axis (Fig.1), the transversal oscillations of the centre of mass do not depend on the longitudinal motion; these oscillations are described by Eq.(1.2) and constitute a special case $(j = 1)$ of the oscillations considered above.

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